

# On Black Hole Stability in Critical Gravities

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## ABSTRACT

We consider extended cosmological gravities with Ricci tensor and scalar squared terms in diverse dimensions. These theories admit solutions of Einstein metrics, including the Schwarzschild-Tangherlini AdS black holes, whose mass and entropy vanish at the critical point. We perform linearized analysis around the black holes and show that in general the spectrum consists of the usual spin-2 massless and ghost massive modes. We demonstrate that there is no exponentially-growing tachyon mode in the black holes. At the critical point, the massless spin-2 modes have zero energy whilst the massive spin-2 modes are replaced by the log modes. There always exist certain linear combination of massless and log modes that has negative energy. Thus the stability of the black holes requires that the log modes to be truncated out by the boundary condition.

Black hole physics is an important research topic in General Relativity. Here an important question is the classical stability of black holes. The simplest black hole is the static and spherically symmetric one, which was first obtained by Schwarzschild. The classical stability of the Schwarzschild black hole was established long ago [1, 2, 3]. The stability condition for static black holes in Einstein gravity with a cosmological constant in higher dimensions was studied in [4]. (See also [5, 6].) In higher dimensions, there exist inhomogeneous Einstein metrics on manifolds that are topologically spheres [7]. The stability of black holes with these level surfaces was discussed in detail in [8].

It is natural to investigate the stability of black holes in extended gravities with higher-derivative curvature terms. One difficulty associated with this line of research is, when polynomials of the Riemann tensor are added to the Lagrangian, it is difficult to construct analytical black hole solutions, even for the static cohomogeneity one configuration. When the coupling of the curvature squared terms are small, perturbative construction were discussed in [9, 10]. Furthermore, the introduction of higher-derivative terms gives rise to ghost-like massive graviton in general [11]. This automatically implies instability of the black hole, even if a solution exists. A counterexample is when higher-derivative terms form a topological Gauss-Bonnet combination. In such case, both problems appear to be resolved. The theory is ghost-free, from the linearized analysis of the vacuum. It can be shown that the propagator of the graviton is of two derivatives rather than four derivatives. Actually, there exists explicit analytical black hole solution in Gauss-Bonnet gravity [12, 13]. However it was shown in [14] that these black holes with small mass are not stable. Moreover, the Gauss-Bonnet term vanishes at  $D = 3$  and is a total derivative in  $D = 4$ . It is only relevant for  $D \geq 5$ .

Recently, critical gravities in four [15] and higher dimensions [16] were obtained, generalizing the results in  $D = 3$  [17, 18]. These theories are extended Einstein gravities with cosmological constants and curvature square terms. These theories admit AdS spacetime as vacuum solutions. Linearized analysis around the vacua shows that for an appropriate choice of parameters, the ghost-like massive graviton is absent. It is replaced by log modes of a fourth-order operator that is the square of the second-order operator of the massless spin-2 graviton. These modes satisfy different asymptotic behavior from the usual massless and massive modes, and they can be consistently truncated out by imposing the standard boundary condition. The explicit forms of log modes were constructed recently [19, 20, 21]. Although an explicit calculation shows that the log modes in four dimensions may have positive energy [15], it was pointed that there are negative norm states associated with

certain linear combination of the massless and log modes [22]. Thus ghost-free condition requires the truncation of log modes by the boundary condition. The purpose of this paper is to generalize the vacuum analysis to static black holes. We find that the critical point derived from black hole backgrounds is the same as that from the vacua. We show that black hole solutions do not suffer from the tachyon instability, but the subtle instability associated with log modes persists, which should hence be truncated out.

Let us consider the action in  $n \geq 3$  dimensions

$$I = \frac{1}{2\kappa^2} \int \sqrt{-g} d^n x [R - (n-2)\Lambda_0 + \alpha R^2 + \beta R^{\mu\nu} R_{\mu\nu}]. \quad (1)$$

Note that the Gauss-Bonnet term is not included, as it is not essential for criticality. In fact, there is no critical phenomenon with purely the Gauss-Bonnet term, since the corresponding propagators of linearized modes are of the second order. In addition, as we remarked earlier, the Gauss-Bonnet term vanishes at  $D = 3$  and it is a total derivative in  $D = 4$ . Furthermore, there is no known analytical black hole solution when we include both the Gauss-Bonnet and other curvature square terms. As we shall see presently, by excluding the Gauss-Bonnet term, these theories admit solutions of Einstein metrics, which include the Schwarzschild-Tangherlini AdS black holes.

The equations of motion are given by

$$\mathcal{G}_{\mu\nu} + \alpha E_{\mu\nu}^{(1)} + \beta E_{\mu\nu}^{(2)} = 0, \quad (2)$$

where

$$\mathcal{G}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \frac{1}{2}(n-2)\Lambda_0 g_{\mu\nu}, \quad (3)$$

$$E_{\mu\nu}^{(1)} = 2R(R_{\mu\nu} - \frac{1}{4}R g_{\mu\nu}) + 2g_{\mu\nu} \square R - 2\nabla_\mu \nabla_\nu R, \quad (4)$$

$$E_{\mu\nu}^{(2)} = -\frac{1}{2}R^{\rho\sigma} R_{\rho\sigma} g_{\mu\nu} + 2R_{\mu\rho\nu\sigma} R^{\rho\sigma} + \square R_{\mu\nu} + \frac{1}{2}\square R g_{\mu\nu} - \nabla_\mu \nabla_\nu R. \quad (5)$$

The trace part of (2) is given by

$$\begin{aligned} & n(n-2)\Lambda_0 - (n-2)R - \alpha \left( (n-4)R^2 - 4(n-1)\square R \right) \\ & - \beta \left( (n-4)R_{\mu\nu} R^{\mu\nu} - n\square R \right) = 0. \end{aligned} \quad (6)$$

Let us consider the Einstein metrics, satisfying

$$R_{\mu\nu} = \Lambda g_{\mu\nu}, \quad R = n\Lambda. \quad (7)$$

The corresponding Einstein tensor is given by

$$\tilde{\mathcal{G}}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \frac{1}{2}(n-2)\Lambda g_{\mu\nu}. \quad (8)$$

The equations of motion reduce to

$$\Lambda_0 = \Lambda + \frac{(n-4)(n\alpha + \beta)}{n-2} \Lambda^2. \quad (9)$$

Thus except in four dimensions, there are two disconnect AdS vacua with different cosmological constants  $\Lambda$ . (In four dimensions, there is only one AdS vacuum whose cosmological constant is the same as the bared  $\Lambda_0$ .) It can be expected that the critical condition depends on the vacuum selected. Had we included the Gauss-Bonnet term in the Lagrangian, the general Einstein metrics could not be the solution, although the theory would still admit AdS vacua, with specified cosmological constants determined by the equations of motion.

The most general ansatz for a spherically symmetric static solution is given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\Omega_d^2, \quad (10)$$

where  $d = n - 2$ . For the metric to be Einstein, we must have  $f = g$ . The most general solution is the Schwarzschild-Tangherlini black hole in the AdS background,

$$f = g = 1 - \frac{\Lambda}{n-1} r^2 - \frac{2m}{r^{d-1}}. \quad (11)$$

The mass of the black hole can be calculated by using the Deser-Tekin method [23],

$$\mathcal{M} = \left(1 + 2\Lambda(n\alpha + \beta)\right) \mathcal{M}_0, \quad (12)$$

where  $\mathcal{M}_0$  is the mass of the same black hole when it is a solution of the Einstein theory without higher derivative terms. It is rather puzzling that the black hole mass, which one expects to be solely determined by the geometry of the spacetime, would depend on the theories which it is embedded in. It is worth remarking that an identical result of the mass can be also obtained from the AMD conformal mass formalism [24, 25]. Using the Wald formula for entropy [26], we have

$$S = \left(1 + 2\Lambda(n\alpha + \beta)\right) S_0, \quad (13)$$

where  $S_0$  is the Bekenstein-Hawking entropy. If the temperature of the black hole is unmodified, the first law of black hole thermodynamics holds in these extended gravities.

In order to study the stability of the above black hole solution, we give the general formalism of the linearized analysis around the background of the Einstein metric (7) with the cosmological constant specified in (9). Varying the metric as  $g_{\mu\nu} \rightarrow g_{\mu\nu} + h_{\mu\nu}$ , and so  $\delta g_{\mu\nu} = h_{\mu\nu}$ , we find to first order in variations that

$$\delta(\mathcal{G}_{\mu\nu} + \alpha E_{\mu\nu}^1 + \beta E_{\mu\nu}^2) = [1 + 2\Lambda(n\alpha + \beta)] \tilde{\mathcal{G}}_{\mu\nu}^L - \beta (\Delta - 2\Lambda) \tilde{\mathcal{G}}_{\mu\nu}^L$$

$$+(2\alpha + \beta) [-\nabla_\mu \nabla_\nu + g_{\mu\nu} \square + \Lambda g_{\mu\nu}] R^L, \quad (14)$$

where  $\tilde{\mathcal{G}}_{\mu\nu}^L$  and  $R^L$  are linearized variations of  $\tilde{\mathcal{G}}_{\mu\nu}$  and  $R$ :

$$\tilde{\mathcal{G}}_{\mu\nu}^L = R_{\mu\nu}^L - \frac{1}{2} R^L g_{\mu\nu} - \Lambda h_{\mu\nu}, \quad (15)$$

$$R_{\mu\nu}^L = \nabla^\lambda \nabla_{(\mu} h_{\nu)\lambda} - \frac{1}{2} \square h_{\mu\nu} - \frac{1}{2} \nabla_\mu \nabla_\nu h, \quad (16)$$

$$R^L = \nabla^\mu \nabla^\nu h_{\mu\nu} - \square h - \Lambda h. \quad (17)$$

We have also defined  $R_{\mu\nu}^L$  as the linearization of  $R_{\mu\nu}$ , and introduced  $h = g^{\mu\nu} h_{\mu\nu}$ . Note that  $\Delta$  is the Lichnerowicz operator, whose action on a two index symmetric tensor  $T_{\mu\nu}$  in the Einstein space is given by

$$\begin{aligned} \Delta T_{\mu\nu} &\equiv -\square T_{\mu\nu} - 2R^\rho{}_{\mu\sigma\nu} T_\rho{}^\sigma + 2R_{(\mu}{}^\rho T_{\nu)\rho} \\ &= -\square T_{\mu\nu} - 2R^\rho{}_{\mu\sigma\nu} T_\rho{}^\sigma + 2\Lambda T_{\mu\nu}. \end{aligned} \quad (18)$$

It is easy to verify that

$$g^{\mu\nu} \tilde{\mathcal{G}}_{\mu\nu}^L = -\frac{1}{2}(n-2)R^L. \quad (19)$$

It follows that the trace part of (14) is given by

$$\left[ \frac{1}{2} (4(n-1)\alpha + n\beta) \square - (n-4)(n\alpha + \beta)\Lambda - \frac{1}{2}(n-2) \right] R^L = 0. \quad (20)$$

In gravity with a cosmological constant, it is convenient to use general coordinate invariance to impose the gauge condition [17]

$$\nabla^\mu h_{\mu\nu} = \nabla_\nu h. \quad (21)$$

It follows from (17) that

$$R^L = -\Lambda h. \quad (22)$$

Substituting this into (20), we see that the spin-0 mode  $h$  is massive with a mass

$$m_0^2 \sim \frac{1}{4(n-1)\alpha + n\beta}. \quad (23)$$

We consider a special case when  $m_0^2$  becomes infinity and hence the spin-0 mode decouples from the spectrum. This corresponds to

$$4(n-1)\alpha + n\beta = 0, \quad (24)$$

for which the equation (20) implies that  $h = 0$ . It is this case, where (24) holds, that we shall focus on in our subsequent discussion. In  $n = 3$ , we have  $8\alpha + 3\beta = 0$ , and the

resulting theory is the new massive gravity in three dimensions [18]. In  $n = 4$ , we have  $3\alpha + \beta = 0$ , and the combination of the curvature squared terms form the Weyl-square term, up to certain total derivatives [15]. In all dimensions, the parameter choice (24) enables one to write the Lagrangian, with an auxiliary field, in the Pauli-Fierz form [21].

Having imposed (24), and hence determined that  $h = 0$ , we are left with a result that the variation of field equations is

$$0 = \delta(\mathcal{G}_{\mu\nu} + \alpha E_{\mu\nu}^1 + \beta E_{\mu\nu}^2) = \frac{2(n-1)\alpha}{n} (\Delta - 2\Lambda)(\Delta - 2\Lambda + M^2) h_{\mu\nu}, \quad (25)$$

where  $h_{\mu\nu}$  is in the transverse traceless gauge

$$\nabla^\mu h_{\mu\nu} = 0, \quad g^{\mu\nu} h_{\mu\nu} = 0, \quad (26)$$

and  $M^2$  is given by

$$M^2 = \frac{(n-2)^2\Lambda}{2(n-1)} + \frac{n}{4(n-1)\alpha}. \quad (27)$$

The most general solution of the fourth-order equation (25) is a linear combination of a massless graviton, satisfying

$$(\Delta - 2\Lambda) h_{\mu\nu}^{(m)} = 0, \quad (28)$$

and a massive spin-2 field, satisfying

$$(\Delta - 2\Lambda + M^2) h_{\mu\nu}^{(M)} = 0. \quad (29)$$

In the AdS background with the cosmological constant  $\Lambda$ , the criterion of stability for spin-2 modes in the background requires that  $M^2 \geq 0$  (see, for example, [27]). Since  $\Lambda$  is negative, we must have

$$0 < \alpha \leq \frac{n}{4(n-2)(-\Lambda)} \quad \longrightarrow \quad \infty > M^2 \geq 0. \quad (30)$$

In particular,  $\alpha$  must be positive. This condition ensures that there is no tachyon instability of the AdS vacuum.

We now examine the stability of the black hole solution given by (10) and (11). The stability of such a solution in standard cosmological Einstein gravities in arbitrary dimensions with a generic Einstein level surface was studied in [4]. It was argued that the most dangerous mode that can produce instability is the transverse traceless tensor mode on the level surface. Since the linearized equation of motion for the massless graviton (28) is exactly the same as that in Einstein gravity, and the equation for the massive graviton is modified by the mass parameter  $M$ , it is reasonable to expect that the dangerous modes

for the tachyon instability are the same as those discussed in [4]. These modes are defined by

$$\begin{aligned} h_{00} &= 0, & h_{0r} &= 0, & h_{0,\alpha} &= 0, & h_{rr} &= 0, & h_{r\alpha} &= 0, \\ h_{\alpha\beta} &= \tilde{h}_{\alpha\beta} r^{\frac{4-d}{2}} \Phi(r) e^{i\omega t}, \end{aligned} \quad (31)$$

where  $\tilde{h}_{\alpha\beta}$  is spherical harmonic in  $d\Omega_d^2$ , satisfying

$$\tilde{\Delta} \tilde{h}_{\alpha\beta} = \lambda \tilde{h}_{\alpha\beta}. \quad (32)$$

The eigenvalues  $\lambda$  were obtained in [28, 29]. They are given by

$$\lambda = (\ell + d - 1)(\ell + 4), \quad (33)$$

with  $\ell = 0, 1, \dots$ . Following the procedure in [4], the function  $\Phi$  satisfies the Schrödinger equation

$$-\frac{d^2 \Phi}{dr_*^2} + V(r(r_*)) \Phi = \omega^2 \Phi. \quad (34)$$

where  $r_*$  is the Regge-Wheeler type of coordinate, defined by

$$dr_* = \frac{dr}{f}. \quad (35)$$

The potential in the coordinate  $r$  is given by

$$V(r) = \frac{\lambda f}{r^2} + \frac{1}{2}(d-4) \frac{f' f}{r} + \frac{1}{4}(d^2 - 10d - 8) \frac{f^2}{r^2} + (M^2 - 2\Lambda) f. \quad (36)$$

For the solution to be absent of the tachyon instability, the potential should ensure that there is no bound state with negative energy  $\omega^2$ . Substituting the black hole solution (11) into the potential, we find that

$$V = \frac{f}{4r^{d+2}} \left( (4M^2 + d^2 + 2d)r^{d+2} + (4\lambda + d^2 - 10d + 8)r^d + d^2 r_+^{d-1} (1 + r_+^2) r \right), \quad (37)$$

where  $r_+$  denotes the horizon of the black hole. For  $M^2 > 0$ , the potential is more positive and hence the solution has better stability. Since we have both  $M^2 > 0$  and  $M^2 = 0$  modes in our case, the criterion for the absence of the tachyon instability is the same as the one given in [4]. In particular, for spherical level surfaces, there is no tachyon instability. The black hole is stable even if the sphere is replaced by other Einstein spaces, as long as the minimum eigenvalue for the tensor harmonic is no less than the following critical value [4]:

$$\lambda_{\text{crit}} \equiv 4 - \frac{(5-d)^2}{4}. \quad (38)$$

This condition is clearly satisfied by spheres. Thus there is no tachyon instability for spherical black holes. The tachyon instability analysis of black holes with the spherical level surfaces replaced by other Einstein metrics is the same as that of the usual gravity studied in [8].

For black holes in higher-derivative gravity, even if the tachyon instability is absent, one still anticipates instabilities associated with the ghost massive graviton. At the linearized level, there is no exponentially-growing behavior associated with the ghost modes; however, their interaction with the normal modes can cause instability, even at the classical level, by transferring energy indefinitely between the normal and ghost sectors.

As we have seen earlier, due to fluctuations of the metric around the Einstein space background, there are massless and massive spin-2 graviton modes. We now examine the on-shell energy of these modes. In the case of AdS background, a procedure for doing this has been first described in [17] for three-dimensional topologically massive gravity, based upon the construction of a Hamiltonian for the graviton field. Generalizations to four and higher dimensions were given in [15] and [16], respectively. For our black hole backgrounds, the procedure is analogous, although it should be modified in order to take into account the fact that the Riemann tensor is no longer trivial. Leaving the parameter  $\alpha$  unrestricted, we may write down the quadratic action whose variation yields the equations of motion (25):

$$\begin{aligned}
I_2 &= -\frac{1}{2\kappa^2} \int \sqrt{-g} d^n x h^{\mu\nu} (\delta\mathcal{G}_{\mu\nu} + \delta E_{\mu\nu}) \\
&= -\frac{(n-1)\alpha}{n\kappa^2} \int \sqrt{-g} d^n x h^{\mu\nu} (\Delta - 2\Lambda)(\Delta - 2\Lambda + M^2) h_{\mu\nu} \\
&= -\frac{(n-1)\alpha}{n\kappa^2} \int \sqrt{-g} d^n x \left[ (\Box h^{\mu\nu})(\Box h_{\mu\nu}) + M^2 (\nabla^\lambda h^{\mu\nu})(\nabla_\lambda h_{\mu\nu}) \right. \\
&\quad \left. - 4 \left( (\nabla^\lambda h_{\mu\nu}) R_{\rho\mu\sigma\nu} (\nabla_\lambda h^{\rho\sigma}) + h^{\mu\nu} (\nabla^\lambda R_{\rho\mu\sigma\nu}) (\nabla_\lambda h^{\rho\sigma}) \right) \right. \\
&\quad \left. + h^{\mu\nu} (4M^2 R_{\rho\mu\sigma\nu} R_{\delta}^{\rho}{}_{\lambda}{}^{\sigma} - 2M^2 R_{\delta\mu\lambda\nu}) h^{\delta\lambda} \right].
\end{aligned} \tag{39}$$

$$\begin{aligned}
&\quad \left. + h^{\mu\nu} (4M^2 R_{\rho\mu\sigma\nu} R_{\delta}^{\rho}{}_{\lambda}{}^{\sigma} - 2M^2 R_{\delta\mu\lambda\nu}) h^{\delta\lambda} \right].
\end{aligned} \tag{40}$$

Using the method of Ostrogradsky for Lagrangians written in terms of second, as well as first, time derivatives we define the conjugate “momenta”

$$\begin{aligned}
\pi^{(1)\mu\nu} &= \frac{\delta L_2}{\delta \dot{h}_{\mu\nu}} - \nabla_0 \left( \frac{\delta L_2}{\delta (d(\nabla_0 h_{\mu\nu})/dt)} \right) \\
&= -\frac{2(n-1)\alpha}{n\kappa^2} \sqrt{-g} \left( M^2 \nabla^0 h^{\mu\nu} - 4 R_{\rho}{}^{\mu}{}_{\sigma}{}^{\nu} \nabla^0 h^{\rho\sigma} - 4 (\nabla^0 R_{\rho}{}^{\mu}{}_{\sigma}{}^{\nu}) h^{\rho\sigma} - \nabla^0 \Box h^{\mu\nu} \right), \\
\pi^{(2)\mu\nu} &= \frac{\delta L_2}{\delta (d(\nabla_0 h_{\mu\nu})/dt)} = -\frac{2(n-1)\alpha}{n\kappa^2} \sqrt{-g} g^{00} \Box h^{\mu\nu}.
\end{aligned} \tag{41}$$

Since the Lagrangian is time-independent, the Hamiltonian is equal to its time average. It is advantageous to writing it in this way, because we can then integrate time derivatives by

parts. Thus we obtain the Hamiltonian

$$\begin{aligned}
H &= T^{-1} \left( \int d^n x \left[ \pi^{(1)\mu\nu} \dot{h}_{\mu\nu} + \pi^{(2)\mu\nu} \frac{\partial(\nabla_0 h_{\mu\nu})}{\partial t} \right] - I_2 \right) \\
&= -\frac{2(n-1)\alpha}{n\kappa^2 T} \int \sqrt{-g} d^n x \left[ M^2 (\nabla^0 h^{\mu\nu}) \dot{h}_{\mu\nu} + 2(\nabla^0 h^{\mu\nu}) \frac{\partial}{\partial t} \left( (\Delta - 2\Lambda) h_{\mu\nu} \right) \right. \\
&\quad \left. - 4(\nabla^0 R_\rho{}^\mu{}_\sigma{}^\nu) h^{\rho\sigma} \dot{h}_{\mu\nu} \right] - \frac{1}{T} I_2,
\end{aligned} \tag{42}$$

where all time integrations are taken over the interval  $T$ . Not all tensor components  $\nabla^0 R_\rho{}^\mu{}_\sigma{}^\nu$  vanish for the general static ansatz (10). However, for the Einstein metrics where  $f = g$ , they all vanish.

Evaluating (42) for the massless mode (satisfying (28)) and for the massive mode (satisfying (29)), we obtain the on-shell energies

$$E_{\text{massless}} = -\frac{2(n-1)\alpha}{n\kappa^2 T} M^2 \int \sqrt{-g} d^n x (\nabla^0 h_{(m)}^{\mu\nu}) \dot{h}_{\mu\nu}^{(m)}, \tag{43}$$

$$E_{\text{massive}} = \frac{2(n-1)\alpha}{n\kappa^2 T} M^2 \int \sqrt{-g} d^n x (\nabla^0 h_{(M)}^{\mu\nu}) \dot{h}_{\mu\nu}^{(M)}. \tag{44}$$

Thus for generic  $M^2 \neq 0$ , ghost excitation is unavoidable. In order for the theory to be free of ghosts, we need to choose the parameter  $\alpha$  such that  $M^2 = 0$ . This is precisely the critical point where the massive spin-2 field also becomes massless. It is of interest to note that the critical condition for black holes is the same as that for the vacuum. Note also that the black hole mass (12) and entropy (13) are both zero at this critical point. The vanishing of black hole masses at critical points was first observed in three dimensions [30], and subsequently in four [15] and general dimensions [16]. The only modes left to discuss are the log modes, which can be obtained from the limit

$$h_{\mu\nu}^{\text{log}} = \lim_{M \rightarrow 0} \frac{h_{\mu\nu}^{(M)} - h_{\mu\nu}^{(m)}}{M^2}. \tag{45}$$

It is clear that the log modes satisfy

$$(\Delta - 2\Lambda) h_{\mu\nu}^{\text{log}} = -h_{\mu\nu}^{(m)}. \tag{46}$$

Thus we see that  $(\Delta - 2\Lambda) h_{\mu\nu}^{\text{log}} \neq 0$  but  $(\Delta - 2\Lambda)^2 h_{\mu\nu}^{\text{log}} = 0$ . Substituting (46) into (42), the on-shell energy for the log modes can be expressed as

$$E_{\text{log}} = \frac{4(n-1)\alpha}{n\kappa^2 T} \int \sqrt{-g} d^n x (\nabla^0 h_{\text{log}}^{\mu\nu}) \dot{h}_{\mu\nu}^{(m)}. \tag{47}$$

We do not expect to find exact analytical solution for these log modes in the black hole background. The log modes in AdS backgrounds were constructed [19, 20, 21]. Their on-shell energies are evaluated in the appendix for  $D = 3$  and  $D = 4$ .

The most general linearized solution at the critical point is thus given by

$$h_{\mu\nu} = c_1 h_{\mu\nu}^{\log} + c_2 h_{\mu\nu}^{(m)}, \quad (48)$$

where  $c_1$  and  $c_2$  are arbitrary constants. We find that the energy is given by

$$\begin{aligned} E(c_1, c_2) &= c_1^2 E_{\log} + 2c_1 c_2 E_{\text{cross}} \\ &= \left( c_1 + \frac{E_{\text{cross}}}{E_{\log}} c_2 \right)^2 E_{\log} - c_2^2 \frac{E_{\text{cross}}^2}{E_{\log}}, \end{aligned} \quad (49)$$

where

$$E_{\text{cross}} \equiv \frac{2(n-1)\alpha}{n\kappa^2 T} \int \sqrt{-g} d^n x (\nabla^0 h_{(m)}^{\mu\nu}) \dot{h}_{\mu\nu}^{(m)}. \quad (50)$$

Note that the cross term  $E_{\text{cross}}$  vanishes only for the “Proca” log modes where  $h_{\mu\nu}^{(m)}$  is pure gauge, *i.e.*  $h_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$ . Thus we see that certain mixed states of the massless and log modes have negative energy. This conclusion, which also applies to the AdS vacua, is equivalent to the analysis of the scalar product of these modes in the AdS background [22].

To summarize, we have considered extended cosmological gravities in diverse dimensions with Ricci scalar and tensor squared terms. These theories were known previously to have critical points where the ghost massive graviton disappears. We showed that these theories admit solutions of Einstein metrics, including both the AdS vacua and Schwarzschild-Tangherlini AdS black holes. We performed linearized analysis around the black hole background and showed that the spectrum consists of the usual spin-2 massless and ghost massive gravitons. We first argued that there is no exponentially-growing tachyon mode in the black holes. Then we examined critical points, where the ghost massive graviton is replaced by the log modes, which have non-vanishing energy, with the sign choice depending on the overall sign of the action. We showed that there always exist certain mixing of the massless and log modes that have negative energy. The stability of the black holes thus requires that the log modes to be truncated out by the boundary condition. (Note that since we have performed the linearized analysis around the generic Einstein metrics, we expect our results apply to the stability of the general Kerr-AdS solutions [31, 32, 33] as well. In fact, it can be checked that the quantities  $\nabla^0 R_{\rho}{}^{\mu}{}_{\sigma}{}^{\nu}$  in (42) vanish for the Kerr-AdS solutions.) This conclusion applies to the stability of the AdS vacua as well. This is different from the critical topologically massive gravity where log modes can be kept in one Virasoro sector, but not the other [34].

Although the boundary condition that removes the log modes may render critical gravities to appear trivial with null modes only, these remaining massless modes are the standard

graviton in the usual Einstein theory of gravity. This is different from the case in three dimensions where the massless modes are indeed pure gauge. Analogously, the non-trivial Schwarzschild-Tangherlini AdS black holes acquire zero mass and entropy in critical gravities. Further investigation of these unusual properties may shed light on the dual log CFT at higher temperature.

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## A Explicit energy of spin-2 modes in $D = 3, 4$ AdS vacua

The massive and massless graviton modes in the  $\text{AdS}_3$  background

$$ds_3^2 = -\cosh^2 \rho dt^2 + \sinh^2 \rho d\phi^2 + d\rho^2 \quad (51)$$

were obtained in [17]. The ansatz is given by

$$\psi_{\mu\nu}(h, \bar{h}) = e^{-i(h+\bar{h})t - i(h-\bar{h})\phi} \frac{\sinh^2 \rho}{(\cosh \rho)^{h+\bar{h}}} \begin{pmatrix} 1 & \frac{1}{2}(h-\bar{h}) & \frac{2i}{\sinh(2\rho)} \\ \frac{1}{2}(h-\bar{h}) & 1 & \frac{i(h-\bar{h})}{\sinh(2\rho)} \\ \frac{2i}{\sinh(2\rho)} & \frac{i(h-\bar{h})}{\sinh(2\rho)} & -\frac{4}{\sinh^2(2\rho)} \end{pmatrix}, \quad (52)$$

where  $h$  and  $\bar{h}$  are constants,  $\psi_{\mu\nu}$  is traceless. The transversality condition is satisfied provided that

$$h - \bar{h} = \pm 2. \quad (53)$$

Let us first consider the upper sign choice, *i.e.*  $\bar{h} = h - 2$ . Then  $\psi_{\mu\nu}$  is a massive graviton satisfying

$$(\square + 2 - M^2)\psi_{\mu\nu} = 0, \quad (54)$$

where

$$M^2 = 4(h-1)(h-2). \quad (55)$$

For  $M^2 > 0$ , we have  $h \leq 1$  or  $h \geq 2$ . The  $h \leq 1$  branch is ruled out by the boundary condition. Thus we must have  $h \geq 2$ , with  $h = 2$  corresponding the massless graviton. For generic  $h$ , we have

$$\begin{aligned} & \int \sqrt{-g} d^3x (\nabla^0 h_{(m)}^{\mu\nu}) \dot{h}_{\mu\nu}^{(m)} \\ &= 2\pi T \int_0^\infty d\rho \frac{4(h-1)(1-2h+\cosh(2\rho)) \sinh \rho}{(\cosh \rho)^{4h+1}}. \end{aligned} \quad (56)$$

For  $h \geq 2$ , the integrand is negative. In particular, the integral is  $-4\pi T/3$  when  $h = 2$ . It follows from (43) and (44) that massless modes have positive energy whilst massive modes have negative energy. In calculating the energy, we have chosen the real part of  $\psi_{\mu\nu}$  as the graviton field. The result is identical if we have chosen the imaginary part.

The log modes, defined by (45), are given by

$$\psi_{\mu\nu}^{\log} = -\frac{1}{2} \left( i t + \log(\cosh \rho) \right) \psi_{\mu\nu}(2, 0). \quad (57)$$

Thus we have

$$\begin{aligned} & \int \sqrt{-g} d^3x (\nabla^0 h_{\log}^{\mu\nu}) \dot{h}_{\mu\nu}^{(m)} \\ &= -4\pi T \int_0^\infty d\rho \tanh \rho \operatorname{sech}^8 \rho \left( 1 + (\cosh(2\rho) - 3) \log(\cosh \rho) \right) \end{aligned} \quad (58)$$

which yields  $-17\pi T/36$ . It follows from (47) that the on-shell energy of log modes is negative for the standard sign choice of the Einstein-Hilbert action, but becomes positive if we reverse the sign of the action. The conclusion is the same for the lower sign choice in (53).

The situation is somewhat different in four dimensions. The log modes was shown to have positive energy in [15]. Here we present some detail. The massive and massless graviton modes in the  $\text{AdS}_4$  background

$$ds_3^2 = -\cosh^2 \rho dt^2 + \sinh^2 \rho (d\theta^2 + \sin^2 \theta d\phi^2) + d\rho^2 \quad (59)$$

were obtained in [21]. They are given by

$$\begin{aligned} \psi_{\tau\tau} &= -\psi_{\tau\phi} = \psi_{\phi\phi} = e^{-3i\tau+2i\phi} \sin^2 \theta \sinh^{-\frac{1}{2}}(2\rho) \tanh^{\frac{5}{2}} \rho, \\ \psi_{\tau\rho} &= -\psi_{\rho\phi} = -i \operatorname{csch} \rho \operatorname{sech} \rho \psi_{\tau\tau}, \\ \psi_{\tau\theta} &= -\psi_{\theta\phi} = i \cot \theta \psi_{\tau\tau}, \quad \psi_{\rho\rho} = -\operatorname{csch}^2(2\rho) \psi_{\tau\tau}, \\ \psi_{\rho\theta} &= -2 \cot \theta \operatorname{csch}(2\rho) \psi_{\tau\tau}, \quad \psi_{\theta\theta} = -\cot^2(\theta) \psi_{\tau\tau}, \end{aligned} \quad (60)$$

They are traceless and transversal and satisfy

$$(\square + 2 - M^2) \psi_{\mu\nu} = 0, \quad \text{with} \quad M^2 = E_0(E_0 - 3). \quad (61)$$

To obtain the on-shell energy, we first need to evaluate the integrand

$$\begin{aligned}
& \int \sqrt{-g} d^4x (\nabla^0 h_{(M)}^{\mu\nu}) \dot{h}_{\mu\nu}^{(M)} \\
&= 2\pi T \int_0^\infty d\rho \left[ -\frac{2}{15} E_0 \left( 68 + 89E_0 + 4(7E_0 - 16) \cosh(2\rho) \right. \right. \\
&\quad \left. \left. + (3E_0 - 4) \cosh(4\rho) \right) \text{sech}^3 \rho (\sinh(2\rho))^{-E_0} (\tanh \rho)^{E_0+2} \right]. \quad (62)
\end{aligned}$$

For  $E_0 \geq 3$ , the integrand is negative. In particular, when the above quantity is  $-9\pi^2 T/16$  when  $E_0 = 3$ . Thus the energy of the massless graviton is positive and that of the massive graviton is negative for non-critical gravity.

The log modes, defined by (45), are given by

$$\psi_{\mu\nu}^{\text{log}} = -\frac{1}{6}(2i t - \log(\sinh(2\rho)) + \log(\tanh \rho)) \psi_{\mu\nu}(E_0 = 3). \quad (63)$$

Thus we have

$$\begin{aligned}
& \int \sqrt{-g} d^4x (\nabla^0 h_{\text{log}}^{\mu\nu}) \dot{h}_{\mu\nu}^{(m)} \\
&= 2\pi T \int_0^\infty d\rho P(\rho), \quad (64)
\end{aligned}$$

where

$$\begin{aligned}
P(\rho) = \frac{1}{120} \tanh^2 \rho \text{sech}^9 \rho \Big( & -2(89 + 28 \cosh(2\rho) + 3 \cosh(4\rho)) \\
& 5(67 + 4 \cosh(2\rho) + \cosh(4\rho))(\log(\sinh(2\rho)) - \log(\tanh(\rho))) \Big). \quad (65)
\end{aligned}$$

This function is non-negative for  $\rho \in [0, \infty)$  and  $\int_0^\infty P d\rho \sim 0.0386968$ . It follows from (47) that the on-shell energy of the log modes is positive for the standard sign of the Einstein-Hilbert term.

However, in general the most general solution is a linear combination of the massless and log modes, as in (48). The corresponding energy is of the form (49). Thus unless we consistently truncate out the log modes, there are always combinations that have negative energies.

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